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Quantum and classical mechanics of q-deformed systems

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Abstract. Quantum and classical mechanics of a system of q-deformed bosonic oscillators are considered. The q-deformed Heisenberg–Weyl algebra of creation and destruction operators is realized by differential operators in a space of functions of real commutative variables. The corresponding Hilbert space is constructed. In this approach, the deformation parameter turns out to be a function of the Planck constant, an oscillator frequency and a parameter with dimension of length. The Hamiltonian path integral is derived and its semiclassical approximation is investigated to obtain the corresponding q-deformed classical theory. The phase space spanned by the usual commutative coordinates is shown to be a cylinder in classical theory. The dimensional parameter introduced determines its radius. It is argued that the q-deformation can be associated with a special non-canonical transformation. The principle of least action for the classical q-deformed system is formulated. A representation of $U_q(m)$ in a space of functions on a phase space spanned by commutative coordinates is constructed.

1. Introduction

The present paper is devoted to quantum and classical mechanics of a system of q-deformed bosonic oscillators. The quantum q-deformed harmonic oscillator is described by creation and destruction operators obeying the non-standard commutation relations [1–3] that depend on the deformation parameter q. When q tends to 1, the commutation relations convert into the standard ones, i.e. they form the Heisenberg–Weyl algebra. The q-deformed oscillators are used to construct representations of quantum groups [4]. However, this dynamical system is interesting by itself and attracts much attention from physicists.

In fact, looking at the history of physics, one can see that physicists have 'deformed' fundamental physical laws several times. A new ('deformed') theory always appears to be more general and contains an initial ('classical') theory as a limit case when the 'deformation' parameter tends to a particular value. For instance, relativistic mechanics becomes Newtonian when the deformation parameter $\beta = v/c$ goes to zero, or quantum mechanics turns into a classical theory in the limit $S/\hbar \to \infty$ (S is an action). Although the deformation parameters v/c and $S/\hbar$ are dimensionless, the physical meaning of the deformations is related to the fundamental dimensional constants c and $\hbar$. These constants determine physical conditions under which the 'deformed' theory becomes the 'classical' one. In other words, the limits $v/c \to 0$ and $S/\hbar \to \infty$ mean that velocities in a system are much less than the light velocity, $v \ll c$, and the action of a system is much greater than the Planck constant, $S \gg \hbar$, respectively.

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It is believed that the so called q-deformation should also be associated with a fundamental dimensional constant, while the deformation parameter q must be a dimensionless function of it and some quantities characterizing a system. In the present work, we realize this programme for an arbitrary system of bosonic q-oscillators.

In section 2, we introduce the basic notation adopted for describing the q-deformed harmonic oscillator and discuss a connection between the ordinary and q-deformed oscillators. We realize the q-deformed Heisenberg–Weyl algebra by differential operators in a space of functions of one real variable. In this approach, the deformation parameter turns out to be a function of the Planck constant, an oscillator frequency and a dimensional parameter l_q (a fundamental length). Then we consider the evolution problem and find a transition amplitude for the q-deformed oscillator. In section 3, we generalize our approach to a system of q-deformed oscillators.

Section 4 is devoted to the path integral approach. We obtain the path integral representation of the transition amplitude for an arbitrary system of q-oscillators. Then we investigate the semiclassical limit of the path integral derived and obtain the corresponding classical theory with ordinary commutative canonical variables. We discover that the phase space of the classical theory is a set of cylinders whose radii are determined by the fundamental length l_q. When l_q tends to infinity, the theory turns into a classical mechanics of usual harmonic oscillators. From the physical point of view, the limit $l_q \to \infty$ means that the energy of each q-deformed oscillator with a frequency ω becomes much lower than the characteristic energy $\omega^2 l_q^2/2$ (see section 5.1).

In section 5, classical Hamiltonian dynamics of q-oscillators is studied. We derive Hamiltonian equations of motion and solve them for some particular Hamiltonians. We analyse the symplectic structure of the theory and show that the q-deformation of classical systems can be associated with a special non-canonical transformation or with a special modification of the standard symplectic structure.

This 'deformed' symplectic structure can be postulated at the very beginning. The canonical quantization of a system of that type leads to the q-deformed quantum mechanics. In this approach, the phase-space variables entering into the effective action in the path integral play the role of the Darboux coordinates for the symplectic structure.

We also formulate the principle of least action for the theory with the 'deformed' symplectic structure. We finish section 5 by constructing a representation of the q-deformed universal enveloping algebra $U_q(m)$ in a space of functions on a phase space spanned by commutative coordinates.

Appendix A contains a brief review of the path integral method for systems with a non-trivial configuration space (boundary conditions). We use results presented in it to elucidate the phase-space structure in a system of q-oscillators.

2. Quantum dynamics of the q-deformed oscillator

The q-deformed oscillator [1–3] is the simplest system in which the creation and destruction operators obey a non-standard algebra depending on a parameter q (the deformation parameter). The limit $q \to 1$ corresponds to the ordinary Heisenberg–Weyl algebra. We consider this system to explain the main points of our approach to the description of q-deformed systems.

2.1. Standard notation

The q-deformed harmonic oscillator is defined by the q-deformed Heisenberg–Weyl algebra [1–3]

$$\hat{a}\hat{a}^{+} - q\hat{a}^{+}\hat{a} = q^{-\hat{N}} \tag{2.1}$$

where an operator $\hat{a}$ and its adjoint $\hat{a}^{+}$ act in an abstract Hilbert space with basis $|n\rangle$, $n = 1, 2, \ldots$, such that

$$\hat{a}\,|n\rangle = [n]_q^{1/2}|n-1\rangle \qquad \hat{a}^{+}|n\rangle = [n+1]_q^{1/2}|n+1\rangle \qquad \hat{a}\,|0\rangle = 0 \tag{2.2}$$

here $[x]_q = (q^x - q^{-x})/(q - q^{-1})$ and q is assumed to be real. The operator $\hat{N}$ plays the role of the number operator

$$\hat{N}\,|n\rangle = n|n\rangle. \tag{2.3}$$

Equality (2.3) comes out from the definitions (2.1) and (2.2). Indeed, one can be convinced that

$$[\hat{a}^{+}, \hat{N}] = -\hat{a}^{+} \qquad [\hat{a}, \hat{N}] = \hat{a} \tag{2.4}$$

which provides the justification of (2.3).

To remove the dependence of the commutation relation (2.1) on the number operator, we carry out the following transformation [3]

$$\hat{b} = q^{\hat{N}/2}\hat{a} \qquad \hat{b}^{+} = \hat{a}^{+}q^{\hat{N}/2}. \tag{2.5}$$

The operators (2.5) satisfy the commutation relation

$$\hat{b}\hat{b}^{+} - q^2\hat{b}^{+}\hat{b} = 1. \tag{2.6}$$

It follows from (2.2), (2.3) and the definition (2.5) that

$$\hat{b}^{+}\hat{b} = (1 - q^{2\hat{N}})/(1 - q^2). \tag{2.7}$$

There is a coordinate realization of the q-deformed harmonic oscillator proposed by Macfarlane [3]

$$\hat{b} = \alpha\left(\mathrm{e}^{-2\mathrm{i}sx} - \mathrm{e}^{-\mathrm{i}sx}\mathrm{e}^{\mathrm{i}s\partial}\right) \qquad \hat{b}^{+} = \alpha^{*}\left(\mathrm{e}^{2\mathrm{i}sx} - \mathrm{e}^{\mathrm{i}s\partial}\mathrm{e}^{\mathrm{i}sx}\right) \tag{2.8}$$

where $\partial \equiv \partial/\partial x$. The commutation relation (2.6) is satisfied if $q = \exp(-s^2)$ and $\alpha\alpha^{*} = (1 - q^2)^{-1}$. We choose q to be positive. The case of negative q corresponds to the choice $q = -\exp(-s^2)$. It changes nothing in what follows. When s runs over the real axis, q varies inside the interval [0, 1]. To derive the theory with $q > 1$, one should replace (x, ∂) by $(-x, \partial)$ in (2.8). Then $q = \exp s^2$. The operator $\hat{b}^{+}$ is the adjoint of $\hat{b}$ in a Hilbert space of functions of one real variable x [3] (see also section 2.3).

2.2. The relation between q-deformed and usual theories

In the limit $q \to 1 (s \to 0)$, the operators (2.8) turn into the destruction and creation operators, respectively, of a usual harmonic oscillator

$$\hat{b} = -\frac{\mathrm{i}}{\sqrt{2}}(x + \partial) + \mathrm{O}(s) \qquad \hat{b}^{+} = \frac{\mathrm{i}}{\sqrt{2}}(x - \partial) + \mathrm{O}(s). \tag{2.9}$$

The real variable x in (2.9) is a coordinate of the configuration space (oscillator amplitude). Comparing (2.9) and (2.8) we conclude that the q-deformation can be associated with a modification of the destruction and creation operators such that x remains as a coordinate spanning the configuration space of the q-deformed system. Therefore, if we believe that q-deformed theories are more general and have to contain the usual ones as limit cases, just as quantum mechanics contains the classical one, then the variable x must be assumed to be a dimensional quantity. This assumption immediately leads to a contradiction. Indeed, the parameter s must also be dimensional, which is impossible because $q = \exp(-s^2)$ is a dimensionless number as follows from (2.6).

Fortunately, (2.8) is not the most general realization. One can change $\exp(\mathrm{i}s\partial) \to \beta\exp(\mathrm{i}s'\partial)$, where β and s' are real, in (2.8), while the commutation relation (2.6) remains untouched if

$$q^2 = \exp(-2ss') \qquad \alpha\alpha^* = (1 - q^2)^{-1}. \tag{2.10}$$

The coefficient β can be eliminated by re-ordering the operators $\exp(\mathrm{i}sx)$ and $\exp(\mathrm{i}s'\partial)$ in (2.8); therefore, we will assume it to be one. If the dimensions of s and s' are opposite, then q is dimensionless.

To specify the constants s and s', we require that the operators $\hat{b}$ and $\hat{b}^{+}$ have the following behaviour in the limit $q \to 1$

$$\hat{b} = \frac{1}{\sqrt{2\omega}}\left(\hat{p} - \mathrm{i}\omega\hat{x}\right) + \mathrm{O}(q-1) \qquad \frac{1}{\sqrt{2\omega}}\left(\hat{p} + \mathrm{i}\omega\hat{x}\right) + \mathrm{O}(q-1) \tag{2.11}$$

where $\hat{p} = -\mathrm{i}\hbar\partial$ and $\hat{x} = x$ (we use the coordinate representation), i.e.

$$[\hat{b}, \hat{b}^{+}] = \hbar + \mathrm{O}(q-1) \tag{2.12}$$

and ω is an oscillator frequency. We also restore the Planck constant. This implies that we put $\hbar$ instead of 1 in the right-hand side of equation (2.6) and, hence, $\alpha\alpha^* = \hbar(1 - q^2)^{-1}$ in (2.8). Our requirement yields the following relations $s' = \gamma s + \mathrm{O}(s^2)$, as $s \to 0$, and $\gamma = \hbar/\omega$.

Introducing a 'fundamental' length $l_q = 1/s$ we obtain the q-deformed quantum theory where the destruction and creation operators [5]

$$\hat{b} = \alpha(\mathrm{e}^{-2\mathrm{i}\hat{x}/l_q} - \mathrm{e}^{-\mathrm{i}\hat{x}/l_q}\mathrm{e}^{-\hat{p}/(\omega l_q)}) \qquad \hat{b}^{+} = \alpha^*(\mathrm{e}^{2\mathrm{i}\hat{x}/l_q} - \mathrm{e}^{-\hat{p}/(\omega l_q)}\mathrm{e}^{\mathrm{i}\hat{x}/l_q}) \tag{2.13}$$

are functions of the standard canonical variables obeying the Heisenberg commutation relation

$$[\hat{x}, \hat{p}] = \mathrm{i}\hbar. \tag{2.14}$$

The parameter of the q-deformation depends on the Planck constant, an oscillator frequency and the fundamental length l_q,

$$q = \exp\left(-\hbar/\omega l_q^2\right) \tag{2.15}$$

so that $q \to 1$ as $l_q \to \infty$. The physical meaning of the dimensional parameter l_q is discussed in sections 2.3 and 4.4.

The dependence of the deformation parameter on the Planck constant is an interesting feature of the approach considered and leads to some consequences. In accordance with the rule of canonical quantization, $[,] = \mathrm{i}\hbar\{,\}$, where $\{,\}$ is the Poisson bracket, we can take the formal classical limit, $\hat{x},\ \hat{p} \to x,\ p$ and $-\mathrm{i}\hbar^{-1}[,] \to \{,\}$ as $\hbar \to 0$. The commutative variables $x,\ p$ play the role of canonically conjugated variables in the classical theory. As $\alpha\alpha^* = \omega l_q^2/2 + \mathrm{O}(\hbar)$, we observe that in the formal classical limit the operators (2.13) become commutative functions of the canonical variables x and p [5, 6]

$$b = l_q\sqrt{\frac{\omega}{2}}\left(\mathrm{e}^{-2\mathrm{i}x/l_q} - \mathrm{e}^{-\mathrm{i}x/l_q}\mathrm{e}^{-p/(\omega l_q)}\right) \qquad b^* = l_q\sqrt{\frac{\omega}{2}}\left(\mathrm{e}^{2\mathrm{i}x/l_q} - \mathrm{e}^{\mathrm{i}x/l_q}\mathrm{e}^{-p/(\omega l_q)}\right). \tag{2.16}$$

This property allows us to construct a 'q-deformed' classical theory of a harmonic oscillator on the ordinary commutative phase space [6] (see section 5). The same result follows from *the semiclassical approximation of the path integral for the transition amplitude* [5], which is discussed in section 4.

2.3. The Hilbert space

In the coordinate representation of (2.13), $\hat{p} = -\mathrm{i}\hbar\partial$, the ground state $\Phi_0(x) = \langle x|0\rangle$ satisfying the equation $\hat{b}\Phi_0 = 0$ reads [3, 5]

$$\Phi_0(x) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{\omega x^2}{2\hbar} + \mathrm{i}\frac{x}{2l_q}\right). \tag{2.17}$$

The excited states $\Phi_n(x) = \langle x|n\rangle$ are determined by applying the operators $\hat{b}^{+n}$ to the ground state (2.17) and have the form

$$\Phi_n(x) = (\alpha^*)^{-n}(\{n\}!)^{-1/2}\hat{b}^{+n}\Phi_0(x) = H_n^q(\exp(2\mathrm{i}x/l_q))\Phi_0(x) \tag{2.18}$$

$$H_n^q(\xi) = (\{n\}!)^{-1/2}\sum_{k=0}^{n}[n,k]_q(-1)^{n-k}q^{n-k}\xi^k \qquad |\xi| = 1 \tag{2.19}$$

where $\{n\}! = \{n\}\{n-1\}\cdots\{1\}$, $\{n\} = 1 - q^{2n}$, and

$$[n,k]_q = \frac{\{n\}!}{\{k\}!\{n-k\}!} \tag{2.20}$$

are the Gaussian polynomials [7]. Using the relation [7] $[n+1,k]_q = [n,k-1]_q + q^{2k}[n,k]_q$ we obtain

$$\hat{b}^+\Phi_n(x) = \alpha^*\{n+1\}^{-1/2}\Phi_{n+1}(x) \qquad \hat{b}\Phi_n(x) = \alpha\{n\}^{-1/2}\Phi_{n-1}(x) \tag{2.21}$$

therefore,

$$b^+b\Phi_n(x) = \alpha\alpha^*\{n\}\Phi_n(x) \tag{2.22}$$

and

$$\hat{N}\Phi_n(x) = n\Phi(x). \tag{2.23}$$

The latter follows from (2.22) and (2.7) (after restoring the Planck constant, the right-hand side of equation (2.7) acquires the factor $\hbar$).

The states (2.18) form an orthogonal basis with respect to the scalar product

$$\langle n|m\rangle = \int_{\Omega_l} \mathrm{d}x\, \sigma_l(x)\Phi_n^*(x)\Phi_m(x) = \delta_{nm} \tag{2.24}$$

where $\Omega_l = [-\pi l_q/2,\ \pi l_q/2]$ and the measure is given by the series

$$\sigma_l(x) = \left(\frac{\hbar}{\pi\omega l_q^2}\right)^{1/2} \sum_{n=-\infty}^{n=\infty} \exp\left[-\frac{\hbar}{\omega l_q^2}\left(n - \frac{\omega l_q}{\hbar}\, x\right)^2\right]. \tag{2.25}$$

The formula (2.24) comes from the normalization relation of the Rogers–Szegoe polynomials $G_n(\theta) = (\{n\}!)^{1/2}(-q)^{-n}H_n^q(-q\exp \mathrm{i}\theta)$ on the unit circle $\theta \in [0,\ 2\pi)$ [3, 7]. The operator $\hat{b}^+$ is the adjoint of $\hat{b}$ in the Hilbert space $\mathcal{H}_q$ of vectors

$$\langle x|\psi\rangle \equiv \psi(x) = \sum_{n=0}^{\infty} \psi_n\Phi_n(x) \tag{2.26}$$

with the scalar product (2.24).

The parameter l_q determines the volume of the physical configuration space, i.e. in this approach, the q-deformation also leads to compactification of the configuration space. In section 4, we prove that the physical configuration space of the q-deformed oscillator is topologically equivalent to a circle S^1. Moreover, the topological structure of the configuration space is preserved in the classical limit, while the corresponding phase space turns out to be a cylinder.

When l_q tends to infinity, the Hilbert space turns into the Hilbert space of an ordinary harmonic oscillator. Indeed, the measure (2.25) can be approximated by the Gaussian integral, $\sum_{n=-\infty}^{\infty} 1/l_q \to \int_{-\infty}^{\infty} \mathrm{d}z,\ z = n/l_q$, as $l_q \to \infty$, which is equal to 1, i.e. $\sigma_l \to 1$. Due to equations (2.24) and (2.11), the functions $\Phi_n(x)$ are converted into the orthonormalized oscillator wavefunctions.

2.4. A transition amplitude

Dynamics of a quantum system is determined by the evolution operator $\hat{U}_t = \exp(-\mathrm{i}t\hat{H}/\hbar)$ where $\hat{H}$ is a Hamiltonian. The matrix element $\langle x|\hat{U}_t|x'\rangle \equiv U_t(x, x')$, where x and x' are points of a configuration space, defines the transition amplitude of a system from an initial point x' to the final one x during a time t. The amplitude satisfies the Schrödinger equation

$$\mathrm{i}\hbar\partial_t U_t(x, x') = \hat{H}(x)U_t(x, x') \tag{2.27}$$

with the initial condition

$$U_{t=0}(x, x') = \langle x|x'\rangle \tag{2.28}$$

where $\langle x|x'\rangle$ is the unit operator kernel.

The evolution problem for q-deformed systems has also been formulated in [8–10]. However, we encounter some uncertainties when applying (2.27) to q-deformed systems. The question is: should the Schrödinger equation be modified under the q-deformation of a quantum system? One of the possible modifications is to change ∂_t by the corresponding q-derivative [10]. At the present time, there is no well founded physical reason to modify the Schrödinger equation; therefore, we will use equation (2.27) when investigating the time evolution of q-deformed systems.

Another question is related to the choice of a q-deformed Hamiltonian. We have no unique method to determine it, having a Hamiltonian of a non-deformed system. One can just impose the requirement that a q-deformed Hamiltonian $\hat{H}_q = H_q(\hat{b}, \hat{b}^+)$ turns into the usual one just as q tends to 1. However, it does not remove any arbitrariness in the choice of $\hat{H}_q$. For instance, for the q-deformed oscillator, one can take

$$\hat{H}_q = \tfrac{1}{2}\omega(\hat{b}^+\hat{b} + \hat{b}\hat{b}^+). \tag{2.29}$$

The corresponding spectrum is determined by (2.7) and (2.23). The choice

$$\hat{H}_q = \hbar\omega(\hat{N} - \tfrac{1}{2}) \tag{2.30}$$

is also available. The Hamiltonians (2.29) and (2.30) have the same limit when $q \to 1$ that coincides with the Hamiltonian of an ordinary oscillator. The Hamiltonian (2.30) possesses a remarkable property: its spectrum is identical to the usual oscillator spectrum. So, the q-deformation does not always modify a spectrum.

For a complete definition of the transition amplitude, one should also describe the properties of a configuration space. It is possible to consider the configuration space of a q-deformed system as an abstract non-commutative space [11]. This point of view on the evolution problem is studied in [10]. We will describe the configuration space, following sections 2.2 and 2.3. Another more general approach is proposed in section 5.2.

Bearing in mind all the reservations mentioned above, we turn directly to the evolution problem for the q-deformed harmonic oscillator. In accordance with the Feynman–Kac formula, the transition amplitude reads

$$U_t^q(x, x') = \sum_{n=0}^{\infty} \Phi_n(x)\Phi_n^*(x')\mathrm{e}^{-\mathrm{i}E_n^q t/\hbar} \tag{2.31}$$

where the spectrum E_n^q depends on the choice of the Hamiltonian $\hat{H}_q$. The equality (2.31) implies that $[\hat{H}_q, \hat{N}] = 0$, i.e. Φ_n are also eigenstates of the q-deformed Hamiltonian. The evolution of a state $\psi(x)$ is determined as follows

$$\psi(x, t) = \int_{\Omega_l} \mathrm{d}x'\, \sigma(x') U_t^q(x, x')\psi(x'). \tag{2.32}$$

The sum (2.31) for the Hamiltonian (2.30) can be transformed to the expression

$$U_t^q(x, x') = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\mathrm{i}\frac{\omega t}{2} - \frac{\omega}{2\hbar}(x^2 + x'^2) + \mathrm{i}\frac{x - x'}{2l_q}\right)$$
$$\times \frac{\{\xi\xi' y^2\}_\infty}{\{y\}_\infty\{\xi y\}_\infty\{\xi' y\}_\infty\{\xi\xi' y\}_\infty} \tag{2.33}$$

where $\xi = -q^{-1}\exp 2ix/l_q$, $\xi' = -q^{-1}\exp(-2ix'/l_q)$, $y = -q\exp i\omega t$ and

$$\{a\}_\infty = \prod_{n=0}^{\infty}(1 - aq^{2n}). \tag{2.34}$$

To obtain (2.33), we substitute the explicit form of Φ_n into (2.31) and use a formula for summation of the Rogers–Szegoe polynomials (see [7] p 50). The product (2.34) is absolutely convergent for $|q|^2 < 1$, $|a| < |q|^{-2}$, which is valid for all the products entering into (2.33) with q defined by (2.15).

The path integral representation for the transition amplitude (2.31) is given in section 4.

3. A system of q-deformed oscillators

A system of m q-deformed oscillators can be described by creation and destruction operators $\hat{a}_i^+$ and $\hat{a}_i$, respectively, satisfying the commutation relations [12]

$$\hat{a}_i\hat{a}_j^+ - q^{\delta_{ij}}\hat{a}_j^+\hat{a}_i = \delta_{ij}q^{-\hat{N}_i} \qquad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^+, \hat{a}_j^+] = 0 \tag{3.1}$$

$$\hat{a}_i^+\hat{a}_i = \frac{q^{\hat{N}_i} - q^{-\hat{N}_i}}{q - q^{-1}} \qquad i, j = 1, 2, \ldots, m. \tag{3.2}$$

The generalization of (2.4) to the case of m oscillators is

$$[\hat{a}_i^+, \hat{N}_j] = -\delta_{ij}\hat{a}_i^+ \qquad [\hat{a}_i, \hat{N}_j] = \delta_{ij}\hat{a}_i. \tag{3.3}$$

Therefore, we can construct an abstract Hilbert space with the basis $|n_1, n_2, \ldots, n_m\rangle \equiv |\boldsymbol{n}\rangle$

$$\hat{a}_i|\boldsymbol{n}\rangle = [n_i]_q^{1/2}|\boldsymbol{n} - 1_i\rangle \qquad \hat{a}_i^+|\boldsymbol{n}\rangle = [n_i + 1]_q^{1/2}|\boldsymbol{n} + 1_i\rangle \tag{3.4}$$

where $|\boldsymbol{n} \pm 1_i\rangle = |n_1, \ldots, n_{i-1}, n_i \pm 1, n_{i+1}, \ldots, n_m\rangle$ and $\hat{a}_i|0\rangle = 0$, so that

$$\hat{N}_i|\boldsymbol{n}\rangle = n_i|\boldsymbol{n}\rangle \qquad n_i = 0, 1, \ldots . \tag{3.5}$$

To obtain the coordinate representation, one should define the operators (2.5) for each degree of freedom

$$\hat{b}_i = \hbar^{1/2}q^{\hat{N}_i/2}\hat{a}_i \qquad \hat{b}_i^+ = \hbar^{1/2}\hat{a}_i^+q^{\hat{N}_i/2} \tag{3.6}$$

which obey the commutation relations

$$\hat{b}_i\hat{b}_j^+ - q^{2\delta_{ij}}\hat{b}_j^+\hat{b}_i = \hbar\delta_{ij} \qquad [\hat{b}_i, \hat{b}_j] = [\hat{b}_i^+, \hat{b}_j^+] = 0. \tag{3.7}$$

The operators (3.6) are the functions of the standard canonical variables [6]

$$\hat{b}_j = \alpha\left(e^{-2i\hat{x}_j/l_q} - e^{-i\hat{x}_j/l_q}e^{-\hat{p}_j/(\omega l_q)}\right) \qquad \hat{b}_j^+ = \alpha^*\left(e^{2i\hat{x}_j/l_q} - e^{-\hat{p}_j/(\omega l_q)}e^{i\hat{x}_j/l_q}\right) \tag{3.8}$$

where

$$\left[\hat{x}_j,\ \hat{p}_k\right] = i\hbar\delta_{jk} \tag{3.9}$$

$\alpha\alpha^* = \hbar(1 - q^2)^{-1}$ and q is defined by (2.15).

In the coordinate representation, $\hat{p}_j = -i\hbar\partial/\partial x_j$, the eigenvectors of the number operators (3.5) are nothing but the product of the functions (2.18) for each degree of freedom

$$\langle x_1, x_2, \ldots, x_m|n_1, n_2, \ldots, n_m\rangle = \prod_{i=1}^{m}\Phi_{n_i}(x_i) \equiv \Phi_{n_1\cdots n_m}(\boldsymbol{x}) \equiv \Phi_{\boldsymbol{n}}(\boldsymbol{x}) = \langle\boldsymbol{x}|\boldsymbol{n}\rangle. \tag{3.10}$$

The states (3.10) form an orthogonal basis with respect the scalar product (2.24) where $\sigma_l(\boldsymbol{x}) = \prod_1^m \sigma_l(x_i)$ and $x_i \in \Omega_l$.

3.1. The $U_q(m)$ symmetry and a choice of the Hamiltonian

A system of m usual non-interacting oscillators with equal frequencies have the $U(m)$ symmetry. This implies that the Hamiltonian commutes with all the generators of $U(m)$. If we wish to keep the q-analogy of the $U(m)$ symmetry after the q-deformation of all oscillators, we should choose a Hamiltonian so that it would commute with all the generators of $U_q(n)$ [13]

$$\hat{e}_{ij} = \hat{a}_i^+ \hat{a}_j \qquad i \neq j = 1, 2, \ldots, m \tag{3.11}$$

$$\hat{d}_i = \hat{N}_i - \hat{N}_{i+1} \qquad i = 1, 2, \ldots, m-1 \tag{3.12}$$

$$\hat{d} = \hat{N} = \sum_{i=1}^{m} \hat{N}_i \tag{3.13}$$

where the operators $\hat{a}_i$ and $\hat{a}_i^+$ obey the algebra (3.1)–(3.2). To obtain the generators in the coordinate representation, one should write them through the operators (3.8). In this case, $\hat{e}_{ij} = \sqrt{q}\hat{b}_i^+ \hat{b}_j q^{\hat{N}_{ij}/2}$, $\hat{N}_{ij} = \hat{N}_i + \hat{N}_j$, while $\hat{d}_i$, $\hat{d}$ keep their form (3.12) and (3.13).

It is easy to see now that the free Hamiltonian $\hat{H} = \hbar\omega \sum \hat{b}_i^+ \hat{b}_i - \hbar m\omega/2$ does not commute with $\hat{e}_{ij}$ and, therefore, there is no system of m non-interacting q-deformed oscillators [13]. The operator $\hat{H}_q = \hbar\omega(\hat{N} - m/2)$ commutes with the generators (3.11)–(3.13) and, in principle, we can take it as the $U_q(m)$-invariant Hamiltonian because it coincides with the free Hamiltonian in the limit $q \to 1$. In this case, only a self-interaction of each oscillator appears due to the non-linearity of the relation (3.2) or (2.7). The spectrum of this Hamiltonian coincides with the spectrum of m free ordinary harmonic oscillators because

$$\hat{H}_q |\boldsymbol{n}\rangle = \hbar\omega \left(\sum_{i=1}^{m} n_i - \frac{m}{2} \right) |\boldsymbol{n}\rangle. \tag{3.14}$$

Due to the absence of the interaction of oscillators, the transition amplitude is factorized

$$U_t^q(\boldsymbol{x}, \boldsymbol{x}') = \prod_{i=1}^{m} U_t^q(x_i, x_i') \tag{3.15}$$

where $U_t^q(x_i, x_i')$ is the transition amplitude for one q-deformed oscillator obtained in section 2.4. So, the usual and q-deformed systems differ from each other only by wavefunctions under this choice of Hamiltonian.

A priori there are no restrictions for choosing the Hamiltonian, except the requirement for its behaviour in the limit $q \to 1$. A non-linear function $\hat{H}_q = H_q(\hat{N})$ can also serve as the $U_q(m)$-invariant Hamiltonian if $\hat{H}_q \to \hat{H}$ as $q \to 1$. The latter leads to an interaction of oscillators. The spectrum becomes more complicated, but its main feature, the degeneracy of each energy level, is kept because of the explicit $U_q(m)$ invariance of the Hamiltonian. The degeneracy for every level is determined by the number of partitions [7] of a fixed eigenvalue $\hat{N} = \sum_1^m n_i$ into positive integers.

4. The path integral approach

The path integral method is a powerful tool for the investigation of quantum dynamics. It gives a natural connection between quantum and classical theories. To obtain a classical theory corresponding to the q-deformed quantum one, we construct the path integral for the transition amplitude and study its semiclassical approximation. In so doing, we find a Hamiltonian and the Poisson bracket of a q-deformed classical theory.

The difficulties of this approach are connected with the compactness of the physical configuration space, $x_i \in \Omega_l$, and the presence of the non-trivial measure $\sigma_l(x)$ in the scalar product (2.24), which prevents us from a direct realization of the standard iterating procedure for the path integral derivation. The compactness of a configuration space is related, as a rule, to some boundary conditions in a quantum theory, for instance a particle in a box [14] or a particle on a circle. A brief consideration of these systems in the framework of the path integral method is given in appendix A to elucidate what type of boundary condition appears in quantum q-deformed systems possessing a Hilbert space with the basis (3.10). To solve the problems mentioned above, we use the method of an analytical continuation of the unit operator kernel $\langle x|x'\rangle$ to the unphysical domain $x_i \in \mathbb{R}$ (see [5, 15, 16]).

4.1. Analytical continuation of the unit operator kernel

We will suppose that a q-deformed quantum theory can be represented as a theory of a set of q-deformed oscillators whose destruction and creation operators obey the commutation relations (3.7). This means that the states (3.10) form a basis in the Hilbert space of the system, but, generally speaking, they are not eigenvectors of a Hamiltonian $\hat{H}_q$ being a function of $\hat{b}_i$ and $\hat{b}_i^+$, $i = 1, 2, \ldots, m$. In principle, one can consider oscillators with different frequencies, i.e. every degree of freedom has a proper deformation parameter $q_i = \exp(-\hbar/(\omega_i l_q^2))$. Some remarks concerning this generalization are given at the end of this subsection.

In accordance with the scalar product (2.24), the unit operator kernel reads

$$\langle x|x'\rangle = \sum_{n_1,\cdots,n_m=0}^{\infty} \Phi_{n_1\cdots n_m}(x)\Phi^*_{n_1\cdots n_m}(x') = \left(\sigma_l(x)\sigma_l(x')\right)^{-1/2}\delta(x-x') \tag{4.1}$$

where x and x' are m-dimensional vectors, elements of $\mathbb{R}^m$, whose coordinates lie in the interval Ω_l, and $\delta(x-x')$ is the m-dimensional delta-function. The analytical continuation of the right-hand side of equation (4.1) is provided by the evident results

$$\Phi_n(x_1, \ldots, x_{j-1}, x_j + \pi l_q, x_{j+1}, \ldots, x_m) = \mathrm{i}\varphi(x_j)\Phi_n(x) \tag{4.2}$$

$$\sigma_l(x_1, \ldots, x_{j-1}, x_j + \pi l_q, x_{j+1}, \ldots, x_m) = \frac{1}{\varphi^2(x_j)}\sigma_l(x) \tag{4.3}$$

$$\varphi(x_j) = \exp\left[-\frac{\pi l_q \omega}{\hbar}\left(x_j + \frac{\pi l_q}{2}\right)\right] \tag{4.4}$$

for each degree of freedom. Equality (4.2) defines the functions $\Phi_n(x)$ over the whole $\mathbb{R}^m$ through their values on $(\otimes\Omega_l)^m$, and, hence, the right-hand side of equation (4.1) can be also continued to $x \in \mathbb{R}^m$. Comparing (4.2) and (4.3), we obtain

$$\langle x|x'\rangle = \frac{\prod_{j=1}^m \exp(\mathrm{i}(x_j - x_j')/2l_q)}{(\sigma_l(x)\sigma_l(x'))^{1/2}} \prod_{j=1}^m \sum_{k_j=-\infty}^{\infty} \delta(x_j - x_j' - \pi k_j l_q) \tag{4.5}$$

$$\equiv \mu_l(x,x')\int_{\mathbb{R}^m} dx''\,\delta(x-x'')Q_q(x'',x') \tag{4.6}$$

$$Q_q(x,x') = \prod_{j=1}^{m}\sum_{k_j=-\infty}^{\infty}\delta(x_j - x'_j - \pi k_j l_q) \equiv \prod_{j=1}^{m} Q_q(x_j - x'_j) \tag{4.7}$$

where $x \in \mathbb{R}^m$ and $x' \in (\otimes\Omega_l)^m$. The kernel (4.5) obeys the quasiperiodical boundary condition (4.2) because of equation (4.3).

For a system of oscillators with different frequencies ω_j ($q \to q_j$ in (3.7)), one should change $\omega \to \omega_j$ in (4.4). Therefore, the final result (4.5) remains unchanged. In what follows, we shall not specially consider the case of different frequencies because the generalization is trivial.

4.2. Derivation of the path integral formula

Before considering the general case of a q-deformed system with m degrees of freedom and an arbitrary Hamiltonian, let us return to the q-deformed harmonic oscillator and derive the path integral representation for this simplest system [5]. To avoid unessential technical complications, we first take the Hamiltonian (2.29). The infinitesimal evolution operator kernel reads

$$U_\epsilon^q(x,x') = \left(1 - \mathrm{i}\frac{\hbar}{\epsilon}\hat{H}_q(x)\right)\langle x|x'\rangle + O(\epsilon^2). \tag{4.8}$$

The unit operator kernel can be written in the integral representation

$$\langle x|x'\rangle = \mu_l(x,x')\int_{-\infty}^{\infty} dx''\int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}e^{\mathrm{i}p(x-x'')/\hbar}Q_q(x'',x'). \tag{4.9}$$

Substituting (4.9) into (4.8) we find

$$U_\epsilon^q(x,x') = \mu_l(x,x')\int_{-\infty}^{\infty} dx''\,\tilde{U}_\epsilon^q(x,x'')Q_q(x'',x') + O(\epsilon^2) \tag{4.10}$$

$$\tilde{U}_\epsilon^q(x,x'') = \int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}\exp\frac{\mathrm{i}}{\hbar}\epsilon\left(p\frac{\Delta}{\epsilon} - \tilde{H}_q(x,p,\Delta)\right) \tag{4.11}$$

$$\tilde{H}_q(x,p,\Delta) = \omega\alpha\alpha^*\Bigg[1 - \frac{\sqrt{q}}{2}\left(q+\frac{1}{q}\right)\left(e^{\mathrm{i}(x-\Delta)/l_q} + e^{-\mathrm{i}x/l_q}\right)e^{-p/(\omega l_q)}$$
$$+\frac{1}{2}\left(1+q^2\right)e^{-\mathrm{i}\Delta/(2l_q)}e^{-2p/(\omega l_q)}\Bigg] \tag{4.12}$$

where $\Delta = x - x''$. For the calculations, we use the following relations (we denote $\hat{D}(x) \equiv \exp(-\hat{p}/(\omega l_q))$, $\hat{p} = -\mathrm{i}\hbar\partial/\partial x$)

$$\hat{D}(x)\sigma_l(x) = \sigma_l\left(x + \frac{\mathrm{i}\hbar}{\omega l_q}\right) = \sigma_l(x) \tag{4.13}$$

$$\hat{D}(x)e^{\mathrm{i}x/l_q}\langle x|x'\rangle = \hat{D}(x)\mu_l(x,x')\int_{-\infty}^{\infty} dx''\,e^{\mathrm{i}x''/l_q}\delta(x-x'')Q_q(x'',x')$$

$$= \mu_l(x,x')\sqrt{q}\int_{-\infty}^{\infty}\frac{dp\,dx''}{2\pi\hbar}e^{i(x-\Delta)/l_q}e^{-p/(\omega l_q)}e^{ip\Delta/\hbar}Q_q(x'',x') \tag{4.14}$$

$$\hat{D}^2(x)\langle x|x'\rangle = (\sigma_l(x)\sigma_l(x'))^{-1/2}\hat{D}^2(x)\int_{-\infty}^{\infty}dx''\,e^{i(x''-x')/(2l_q)}\delta(x-x'')Q_q(x'',x')$$

$$= \mu_l(x,x')\int_{-\infty}^{\infty}\frac{dp\,dx''}{2\pi\hbar}e^{-i\Delta/(2l_q)}e^{-2p/(\omega l_q)}e^{ip\Delta/\hbar}Q_q(x'',x'). \tag{4.15}$$

The transition amplitude for a finite time t is defined as the limit of the convolution of N infinitesimal kernels (4.10)

$$\int_{\Omega_l}\prod_{i=1}^{N}(dx_i\,\sigma_l(x_i))\,U_\epsilon^q(x,x_1)U_\epsilon^q(x_1,x_2)\cdots U_\epsilon^q(x_N,x') \to U_t^q(x,x') \tag{4.16}$$

as $N\to\infty$, $\epsilon\to 0$ so that $t=(N+1)\epsilon$ is fixed and $\Delta/\epsilon = \dot{x}+O(\epsilon)$. The finite-dimensional approximation (4.16) contains integrations over the interval Ω_l. To transform the path integral measure in (4.16) to the standard one with infinite integration limits, we calculate explicitly the convolution of two infinitesimal kernels (4.10),

$$U_{2\epsilon}^q(x,x') = \int_{\Omega_l}dx_1\,\sigma_l(x_1)U_\epsilon^q(x,x_1)U_\epsilon^q(x_1,x') \tag{4.17}$$

$$= \mu_l(x,x')\int_{-\infty}^{\infty}dx_2\,\tilde{U}_\epsilon^q(x,x_2)\int_{\Omega_l}dx_1\,Q_q(x_2,x_1)$$
$$\times\int_{-\infty}^{\infty}dx''\,\tilde{U}_\epsilon^q(x_1,x'')Q_q(x'',x'). \tag{4.18}$$

The integral over x_1 can be taken with the formula

$$\int_{\Omega_l}dx'\,Q_q(x,x')f(x') = \sum_{k=-\infty}^{\infty}\Theta_{\Omega_l}(x-k\pi l_q)f(x-k\pi l_q) \equiv f_Q(x) \tag{4.19}$$

where $\Theta_{\Omega_l}(x)$ is the characteristic function of the region Ω_l, $\Theta_{\Omega_l}(x)=1$ for $x\in\Omega_l$ and it vanishes outside Ω_l. For periodical functions, $f(x+\pi l_q)=f(x)$, we have $f_Q(x)=f(x)$. If $f(x+\pi l_q)=-f(x)$, then $f_Q(x)=\varepsilon_l(x)f(x)$ where

$$\varepsilon_l(x) = \sum_{k=-\infty}^{\infty}(-1)^k\Theta_{\Omega_l}(x-\pi k l_q) = -\varepsilon_l(x+\pi l_q). \tag{4.20}$$

Thus, $f_Q(x)$ is a periodical continuation of a function $f(x)$ outside the interval Ω_l, $f_Q(x+\pi l_q)=f_Q(x)$.

The kernel $\tilde{U}_\epsilon^q(x_1,x'')$ in (4.18) depends on $\exp(\pm ix_1/l_q)$ and $\Delta = x_1-x''$ (see (4.11) and (4.12)). After integration over x_1 in (4.18), these quantities change as follows

$$\exp(\pm ix_1/l_q)\to\varepsilon_l(x_2)\exp(\pm ix_2/l_q) \tag{4.21}$$

and $\Delta\to x_2-x''-k\pi l_q\equiv\Delta-k\pi l_q$ in accordance with (4.19). The shift of Δ can be removed by changing the integration variable in (4.18), $x''\to x''-k\pi l_q$. So, the dependence of $\tilde{U}_\epsilon^q(x_2,x'')$ on Δ remains untouched after integration over x_1. As a result,

$$U_{2\epsilon}^q(x,x') = \mu_l(x,x')\int_{-\infty}^{\infty}dx''\left(\int_{-\infty}^{\infty}dx_2\tilde{U}_\epsilon^q(x,x_2)\tilde{U}_\epsilon^q(x_2,x'')\right)Q_q(x'',x') \tag{4.22}$$

where the replacement (4.21) in $\tilde{U}^q_\epsilon(x_2, x'')$ is assumed.

One can easily transform the measure in (4.16) to the standard phase-space path integral measure with the help of (4.22). Carrying out this we obtain the phase-space path integral for the transition amplitude of the q-deformed harmonic oscillator

$$U^q_t(x, x') = \mu_l(x, x') \int_{-\infty}^{\infty} \mathrm{d}x'' \, \tilde{U}^q_t(x, x'') Q_q(x'', x') \tag{4.23}$$

$$\tilde{U}^q_t(x, x') = \int_{-\infty}^{\infty} \prod_{\tau=0}^{t} \left(\frac{\mathrm{d}p(\tau)\, \mathrm{d}x(\tau)}{2\pi\hbar} \right) \exp \frac{\mathrm{i}}{\hbar} \int_{\tau}^{t} \mathrm{d}\tau \left[p\dot{x} - H_l(x, p, \hbar) \right] \tag{4.24}$$

$$H_l(x, p, \hbar) = \hbar\omega \frac{1+q^2}{1-q^2} \left[\frac{1}{1+q^2} - \frac{1}{\sqrt{q}} \sqrt{1 - \sin^2(x/l_q)} \; D(p) + \frac{1}{2} D^2(p) \right] \tag{4.25}$$

where $D(p) = \exp(-p/(\omega l_q))$ and q is given by (2.15). The square root in the second term of (4.25) (or the absolute value of $\cos(x/l_q)$) appears because of the change (4.21) in (4.22) and, hence, in (4.16). The dependence of the path integral on the kernel Q_q defined by (4.7) means that the configuration space of the system is topologically equivalent to a circle S^1. The influence of topology of a configuration space (boundary conditions) on the path integral is briefly discussed in appendix A. In fact, there is a unique correspondence between a form of the operator $\hat{Q}_q$ and a type of boundary conditions in a quantum theory [15, 16]. Comparing the kernels (4.7) and (6.10) (see appendix A, a particle on a circle) we conclude that the configuration space of the q-deformed harmonic oscillator is a circle with the radius $l_q/2$. In the limit $l_q \to \infty$, i.e. when the radius of the configuration space tends to infinity, we recover quantum mechanics of the ordinary bosonic oscillator since $\mu_l(x, x') \to 1$, $Q_q(x, x') \to \delta(x - x')$ and $H_l(x, p, \hbar) \to H(x, p) = (p^2 + x^2)/2$. Thus, $U^q_t(x, x')$ turns into the path integral for the ordinary harmonic oscillator when $q \to 1$.

4.3. The case of an arbitrary Hamiltonian

All the calculations from the previous subsection can be repeated for an arbitrary Hamiltonian $\hat{H}_q = H_q(\hat{b}, \hat{b}^+)$. We will assume the Hamiltonian to be a polynomial of $\hat{b}$ and $\hat{b}^+$. Therefore, it can be written in the following Hermitian form

$$\hat{H}_q(x) = \sum_{n,n'} \left(\beta_{nn'} \hat{D}^n(x) \mathrm{e}^{\mathrm{i}n'x/l_q} + \beta^*_{nn'} \mathrm{e}^{-\mathrm{i}n'x/l_q} \hat{D}^n(x) \right) \tag{4.26}$$

where $\beta_{nn'}$ are coefficients depending on q. To obtain the relations (4.10)–(4.12) in the general case, one should use the following formula

$$\begin{aligned} \hat{H}_q(x)\langle x|x'\rangle &= \hat{H}_q(x)(\sigma_l(x)\sigma_l(x'))^{-1/2} \int_{-\infty}^{\infty} \mathrm{d}x'' \, \mathrm{e}^{\mathrm{i}(x''-x')/(2l_q)} \delta(x - x'') Q_q(x'', x') \\ &= \mu_l(x, x') \int_{-\infty}^{\infty} \frac{\mathrm{d}p\, \mathrm{d}x''}{2\pi\hbar} \tilde{H}_q(x, p, \Delta) \mathrm{e}^{\mathrm{i}p\Delta/\hbar} Q_q(x'', x') \end{aligned} \tag{4.27}$$

where $\Delta = x - x''$ and

$$\tilde{H}_q(x, p, \Delta) = \sum_{n,n'} \left(\beta_{nn'} \mathrm{e}^{\mathrm{i}n'(x-\Delta)/l_q} + \beta^*_{nn'} \mathrm{e}^{-\mathrm{i}n'x/l_q} \right) D^n(p) \mathrm{e}^{-\mathrm{i}\Delta/(2l_q)}. \tag{4.28}$$

The convolution (4.16) can be calculated in a similar way (see (4.17)–(4.22)). Taking the limit $N \to \infty$, $\epsilon \to 0$, $\Delta/\epsilon \to \dot{x}$ in the finite-dimensional approximation of the path integral and neglecting terms of the order ϵ^2 in the exponential, we arrive at the final formulae (4.23) and (4.24) where

$$H_l(x, p, \hbar) = \sum_{n,n'} \left(\beta_{nn'} e^{in'x/l_q} + \beta^*_{nn'} e^{-in'x/l_q}\right) D^n(p)\varepsilon_l^{n'}(x). \tag{4.29}$$

By the definition (4.20), $\varepsilon_l^n(x) = 1$ for even n and it is equal to $\varepsilon_l(x)$ for odd n.

Thus, the structure of the operator $\hat{Q}_q$ is independent of the choice of the Hamiltonian. We conclude that the phenomenon of compactification of the configuration space takes place for an arbitrary Hamiltonian.

Let us consider briefly the case of m degrees of freedom. We shall not give explicit calculations because they differ from the ones carried out above just by replacing $n \to \boldsymbol{n}$ and $x \to \boldsymbol{x}$. Instead of (4.26), one should take the Hamiltonian

$$\hat{H}_q(\boldsymbol{x}) = \sum_{\boldsymbol{n},\boldsymbol{n}'} \left(\beta_{\boldsymbol{n}\boldsymbol{n}'} \prod_{j=1}^{m} \hat{D}^{n_j}(x_j) e^{in'_j x_j/l_q} + \beta^*_{\boldsymbol{n}\boldsymbol{n}'} \prod_{j=1}^{m} e^{-in'_j x_j/l_q} \hat{D}^{n_j}(x_j)\right). \tag{4.30}$$

Repeating the calculations (4.27)–(4.28) we obtain the path integral representation for the transition amplitude. It has the form (4.23) where x, x' and x'' should be replaced by $\boldsymbol{x}, \boldsymbol{x}'$ and $\boldsymbol{x}''$, respectively; also $p\dot{x} \to (\boldsymbol{p}, \dot{\boldsymbol{x}})$ and $(2\pi\hbar)^{-1} dp(\tau)dx(\tau) \to (2\pi\hbar)^{-m} d\boldsymbol{p}(\tau)\, d\boldsymbol{x}(\tau)$ in (4.24); the effective Hamiltonian $H_l(\boldsymbol{x}, \boldsymbol{p}, \hbar)$ is obtained from (4.30) by changing $\hat{D}^{n_j}(x_j) \to D^{n_j}(p_j)\varepsilon_l^{n'_j}(x_j)$. Due to the factorization of the operator $\hat{Q}_q$ (see (4.7)), we discover that the topology of the configuration space in the case of m degrees of freedom is $(\otimes S^1)^m$.

4.4. A semiclassical approximation and q-deformed classical mechanics

The effective Hamiltonian $H_l(\boldsymbol{x}, \boldsymbol{p}, \hbar)$ depends on the Planck constant because the coefficients $\beta_{\boldsymbol{n}\boldsymbol{n}'}$ are functions of the deformation parameter q (or q_j). Hence, H_l cannot serve as a Hamiltonian of a classical theory. The dependence of H_l on $\hbar$ is just due to the operator ordering in the quantum Hamiltonian. Indeed, consider a system with one degree of freedom (a generalization is trivial). The operator $\hat{H}_q$ contains terms like $\prod_j (\hat{b}^{+n_j}\hat{b}^{n'_j})$ where $j < \infty$. To obtain the representation (4.26), we have to order the operators $\hat{D}(x)$ and $\exp(ix/l_q)$ in the product of $\hat{b}^+$ and $\hat{b}$. In contrast with Hamiltonians bilinear in canonical momenta where operator ordering terms are additive, the operator ordering corrections in $\hat{H}_q$ are multiplicative due to the following relation

$$\gamma_{nn'}(\alpha, \alpha^*) \prod_j \left(\hat{D}^{n_j}(x)\ e^{in'_j x/l_q}\right) = \gamma_{nn'}(\alpha, \alpha^*) q^{\lambda(n,n')} \hat{D}^n(x) e^{in'x/l_q} \tag{4.31}$$

here $n = \sum_j n_j$, $n' = \sum_j n'_j$ and

$$\lambda(n, n') = \sum_j n_j \left(\sum_{j'<j} n'_{j'}\right). \tag{4.32}$$

The equality (4.31) defines the dependence of $\beta_{nn'}$ on $\hbar$,

$$\beta_{nn'}(\alpha, \alpha^*, q) = \gamma_{nn'}(\alpha, \alpha^*) q^{\lambda(n,n')}. \tag{4.33}$$

The coefficients $\gamma_{nn'}$ are polynomials of α and α^* since $\hat{H}_q$ is assumed to be a polynomial of the destruction and creation operators.

It follows from equation (4.33) that $\beta_{nn'}$ are regular functions of the Planck constant and can be decomposed into series over powers of $\hbar$ because the module of α has also a regular behaviour when $\hbar \to 0$.

$$\alpha\alpha^* = \frac{\hbar}{1-q^2} = \frac{\omega l_q^2}{2} + \mathrm{O}(\hbar). \tag{4.34}$$

Using (4.33) and (4.34) we have

$$H_l(x, p, \hbar) = H_q(b^*, b) + \mathrm{O}(\hbar) \tag{4.35}$$

where b^* and b are defined in (2.16) and $H_q(b^*, b)$ plays the role of the classical Hamiltonian. The second term in (4.35) vanishes when $\hbar \to 0$ and, therefore, has to be identified with quantum corrections to the classical Hamiltonian $H_q(b^*, b)$. Quantum corrections in the effective action entering into a path integral always occur through the operator ordering, which actually means that the path integral depends on the operator ordering [15].

The measure $\mu_l(x, x')$ can also be associated with the quantum corrections because $\mu_l = 1+\mathrm{O}(\hbar)$. In contrast with μ_l, the operator $\hat{Q}_q$ in the path integral (4.23) is independent of the Planck constant and, hence, the compactification of the configuration space of q-deformed systems must take place in the classical limit. For this reason we can omit the function $\varepsilon_l(x)$ in the classical limit of (4.29) because $\varepsilon_l(x) = 1$ if $x \in \Omega_l$, i.e. $\varepsilon_l(x) = 1$ on the physical configuration space $x \in S^1$.

In the general case of a q-deformed system with m degrees of freedom and the Hamiltonian $\hat{H}_q = \hat{H}_q(\hat{\boldsymbol{b}}^+, \hat{\boldsymbol{b}})$, $\hat{\boldsymbol{b}}$ means the set $(\hat{b}_j)$, $j = 1, 2, \ldots, m$, one can also obtain the corresponding classical theory with the Hamiltonian

$$H_l(\boldsymbol{x}, \boldsymbol{p}) \equiv H_l(\boldsymbol{x}, \boldsymbol{p}, \hbar = 0) = H_q(\boldsymbol{b}^*, \boldsymbol{b}) \tag{4.36}$$

which is defined on a phase space with the ordinary symplectic form (the Poisson bracket of canonical variables)

$$\left\{x_j,\ p_k\right\} = \delta_{jk}. \tag{4.37}$$

The quantities $\boldsymbol{b}^*$ and $\boldsymbol{b}$ can be called the q-deformed phase-space holomorphic variables. In the limit $l_q \to \infty$, they convert into the ordinary ones,

$$b_j = (p_j - \mathrm{i}\omega_j x_j)/\sqrt{2\omega_j} + \mathrm{O}(1/l_q) \qquad b_j^* = (p_j + \mathrm{i}\omega_j x_j)/\sqrt{2\omega_j} + \mathrm{O}(1/l_q). \tag{4.38}$$

The existence of the standard symplectic structure in the classical theory allows us to recover the q-deformed quantum theory by means of the canonical quantization when all the canonical variables x_j and p_j become operators with the commutation relation induced by the Poisson bracket $[,] = \mathrm{i}\hbar\{,\}$. Let us verify this for the classical theory obtained above. Changing the canonical variables p_j and x_j by the operators $\hat{p}_j$ and $\hat{x}_j$ in (2.16) ($\omega \to \omega_j$) for each degree of freedom, we get the operators $\hat{c}_j$ and $\hat{c}_j^+$ that satisfy the relation †

$$\hat{c}_i\hat{c}_j^+ - q_i^{2\delta_{ij}}\hat{c}_j^+\hat{c}_i = \tfrac{1}{2}\omega_i l_q^2\left(1 - q_i^2\right)\delta_{ij} \tag{4.39}$$

† The operator ordering problem can be solved by imposing the requirement $(\hat{c}_j^+)^+ = \hat{c}_j$ if $\hat{x}^+ = \hat{x}$ and $\hat{p}^+ = \hat{p}$.

where q_i is given by (2.15) with $\omega = \omega_i$. Setting

$$\hat{c}_i = \beta \hat{b}_i \qquad \hat{c}_i^+ = \beta_i^* \hat{b}_i^+ \qquad \beta_i \beta_i^* = \frac{\omega_i l_q^2}{2\hbar}\left(1 - q_i^2\right) \tag{4.40}$$

we obtain the quantum theory with the destruction and creation operators obeying the q-deformed Heisenberg–Weyl algebra (3.7).

This simple observation shows us that the canonical quantization and so called q-deformation can be independently done for a classical theory, also the limits $q \to 1$ and $\hbar \to 0$ commute with each other. Thus, taking first the classical limit $\hbar \to 0$ and then the limit $l_q \to \infty$ in the q-deformed quantum theory, we arrive at a classical theory; the same classical theory can be recovered by taking first the limit $l_q \to \infty$ and then the classical limit $\hbar \to 0$. Going in the opposite directions from a classical theory, quantization $\to$ q-deformation and q-deformation $\to$ quantization, we obtain the same q-deformed quantum theory.

Consider the classical Hamiltonian (4.35) for the harmonic q-oscillator. Taking the limit $\hbar \to 0$ in (4.25) we find

$$H_l(x, p, \hbar = 0) = \tfrac{1}{2}\omega^2 l_q^2 [(D(p) - |\cos(x/l_q)|)^2 + \sin^2(x/l_q)]. \tag{4.41}$$

An interesting feature of the theory with the Hamiltonian (4.41) is that it has a degenerate classical 'vacuum' state, i.e. if we treat (4.41) as an energy of the system, then its absolute minimum is provided by the phase-space configurations $p = 0$, $x = \pi l_q n$ with n being a number. This resembles a particle in a periodic potential (see [17] and references therein). In quantum theory, the ground state of the particle splits into a zone due to tunnelling. Energy levels in the zone are numerated by a continuous parameter θ, i.e. there appears a θ-vacuum structure. In quantum theory of the q-deformed harmonic oscillator, we have no θ-vacuum structure in spite of the degeneracy of the classical ground state. Notice that the states (2.18) are exact eigenstates of the Hamiltonian (2.29) (see (4.29)), therefore, there is no θ-zone around the ground state of the q-deformed harmonic oscillator.

It seems, however, that another choice of the Hamiltonian (see (2.22)) could provide the θ-vacuum structure. We shall not investigate this problem herein because principles for choosing Hamiltonians of q-deformed systems remain far from obvious.

5. Classical mechanics of q-deformed systems

5.1. Solutions to classical equations of motion

Let us calculate the Poisson bracket of the q-deformed holomorphic variables b_j and b_j^*. Using (4.37) we have [6]

$$\{b_i,\ b_j^*\} = -\mathrm{i}\,\delta_{ij}\left(1 - \frac{\omega_j b_j^* b_j}{E_j^{(l)}}\right) \qquad E_i^{(l)} = \frac{\omega_i^2 l_q^2}{2}. \tag{5.1}$$

Assuming a Hamiltonian to be a function of the holomorphic variables, $H_q = H_q(\boldsymbol{b}, \boldsymbol{b}^*)$, we obtain the Hamiltonian equations of motion

$$\dot{b}_j = \{b_j,\ H_q\} = -\mathrm{i}\left(1 - \frac{\omega_j b_j^* b_j}{E_j^{(l)}}\right)\frac{\partial H_q}{\partial b_j^*} \tag{5.2}$$

$$\dot{b}_j^* = \{b_j^*,\ H_q\} = \mathrm{i}\left(1 - \frac{\omega_j b_j^* b_j}{E_j^{(l)}}\right)\frac{\partial H_q}{\partial b_j}. \tag{5.3}$$

Equations (5.2) and (5.3) can be integrated for certain particular Hamiltonians. Let, for instance, H_q be a function of m variables $h_i = \omega_i b_i^* b_i$ (the energy of a free q-oscillator with a frequency ω_i). Obviously, $\{h_i,\ h_j\} = 0$ and , therefore, they are integrals of motion, $h_i = E_i =$ constant. The equations of motion are simplified

$$\dot{b}_j = -\mathrm{i}\Omega_j b_j \tag{5.4}$$

$$\dot{b}_j^* = \mathrm{i}\Omega_j b_j^* \tag{5.5}$$

$$\Omega_j = \omega_j \left(1 - \frac{E_j}{E_j^{(l)}}\right) \frac{\partial E}{\partial E_j} \tag{5.6}$$

where $E = H_q(E_1, \ldots, E_m)$ is the energy of the system. Thus, we observe that the holomorphic variables oscillate with the frequencies depending on the energy of the oscillators. The evolution of the canonical variables $\boldsymbol{x}, \boldsymbol{p}$ can be found from the equalities

$$\exp\left(\frac{2\mathrm{i}x_j}{l_q}\right) = \frac{1 + \gamma_j b_j^*}{1 + \gamma_j b_j} \tag{5.7}$$

$$\exp\left(-\frac{2p_j}{\omega_j l_q}\right) = \left(\frac{1 + \gamma_j b_j}{1 + \gamma_j b_j^*} - \gamma_j b_j\right)\left(\frac{1 + \gamma_j b_j^*}{1 + \gamma_j b_j} - \gamma_j b_j^*\right) \tag{5.8}$$

where $\gamma_j = \sqrt{\omega_j / E_j^{(l)}}$. We remember that $x_j(t) \in S^1$, i.e. the configurations $x_j(t)$ and $x_j(t) + \pi l_q$ are related to the same physical state and must be identified.

In the formal limit $l_q \to \infty$, the Poisson bracket (5.1) turns into the standard one. Actually, l_q is a fundamental constants like $\hbar$ and, therefore, it cannot 'tend' to 'anything'. The physical meaning of the limit $l_q \to \infty$ is that the q-deformed system becomes the classical one when the energy of a free q-oscillator is much less than the characteristic energy $E_i^{(l)}$, $E_i \ll E_i^{(l)}$. This resembles the relation between quantum and classical mechanics when the formal limit $\hbar \to 0$ means that the action S of a system is much greater than the Planck constant, $S \gg \hbar$.

5.2. *The q-deformation and non-canonical transformations*

The equalities (2.16) determine a non-canonical transformation $x_j,\ p_j \to X_j,\ P_j$,

$$X_j = X_j(\boldsymbol{x}, \boldsymbol{p}) = \frac{\mathrm{i}}{\sqrt{2\omega_j}}\left(b_j - b_j^*\right) \qquad P_j = P_j(\boldsymbol{x}, \boldsymbol{p}) = \sqrt{\frac{\omega_j}{2}}\left(b_j + b_j^*\right) \tag{5.9}$$

$$\{X_i,\ P_j\} = \delta_{ij}\left[1 - \frac{1}{2E_j^{(l)}}\left(P_j^2 + \omega_j^2 X_j^2\right)\right] \tag{5.10}$$

where we use the relation (5.1). Therefore, the q-deformation can be identified with the non-canonical transformation (5.9). In fact, the variables $\boldsymbol{x}, \boldsymbol{p}$ play the role of the Darboux variables [18] for the symplectic structure (5.10). We remember that, by definition, the symplectic structure in the Darboux variables has the standard form (4.37) [18].

The symplectic structure (5.10) on the phase space spanned by the coordinates $\boldsymbol{X}, \boldsymbol{P}$ can be postulated at the very beginning. It is easy to be convinced that the Poisson bracket thus defined satisfies all the characteristic properties [18]—bilinearity, skew-symmetry, the Leibnitz rule and the Jacobi identity. The canonical quantization of a classical system with the 'q-deformed' symplectic structure (5.10) leads to the non-standard Heisenberg commutation relation [9]

$$\left[\hat{X}_j,\ \hat{P}_k\right] = \mathrm{i}\hbar\delta_{jk}\left[1 - \frac{1}{2E_j^{(l)}}\left(\hat{P}_j^2 + \omega_j^2\hat{X}_j^2\right)\right]. \tag{5.11}$$

It is known that non-canonical commutators for position and momentum operators [19] can lead to q-deformed quantum theories [9, 20, 21]. Indeed, introducing the operators $\hat{b}_j$ and $\hat{b}_j^+$ connected with $\hat{X}_j$ and $\hat{P}_j$ by the relations (5.9), we have

$$\hat{b}_j\hat{b}_k^+ - \hat{b}_k^+\hat{b}_j = \hbar\delta_{jk}\left(1 - \frac{\omega_j}{2E_j^{(l)}}\left(\hat{b}_j\hat{b}_j^+ + \hat{b}_j^+\hat{b}_j\right)\right). \tag{5.12}$$

Renormalizing the creation and destruction operators $\hat{b}_j \to (1 + \hbar\omega_j/(2E_j^{(l)}))^{-1/2}\hat{b}_j$ (analogously for $\hat{b}_j^+$) we obtain

$$\hat{b}_j\hat{b}_k^+ - q_j^{\delta_{jk}}\hat{b}_k^+\hat{b}_j = \hbar\delta_{jk} \tag{5.13}$$

$$q_j = \frac{1-\gamma_j}{1+\gamma_j} \qquad \gamma_j = \frac{\hbar\omega_j}{2E_j^{(l)}}. \tag{5.14}$$

The deformation parameter in this approach differs from (2.15).

To obtain a relation between the theory (5.11) and quantum mechanics constructed in section 2, one should realize the operators $\hat{X}_j$ and $\hat{P}_j$ in a space of functions of m real variables, i.e. introduce a representation like (3.8). In so doing, we find that the dimensional parameter l_q is not a fundamental constant in quantum theory; it turns out to be a function of the Planck constant and parameters $E_j^{(l)}$ and ω_j characterizing the classical theory (5.10). We observe also that l_q can depend on the number j of degrees of freedom. Indeed, comparing (5.13), (5.14) and (3.7), (2.15) we find the relation

$$\exp\left(-\frac{2\hbar}{\omega_j l_q^2}\right) = \frac{1-\gamma_j}{1+\gamma_j}. \tag{5.15}$$

Hence,

$$l_q^2 = l_q^2(\hbar, E_j^{(l)}, \omega_j) = -\frac{2\hbar}{\omega_j(\ln(1-\gamma_j) - \ln(1+\gamma_j))}. \tag{5.16}$$

As has been shown above, the parameter l_q determines the 'volume' of the physical configuration space in the quantum theory and therefore this volume becomes a function of the Planck constant . Moreover, the operator $\hat{Q}_q$ entering into the path integral (4.23) also depends on the Planck constant. This gives rise to a natural question: what happens in the semiclassical limit then? It is easy to see that

$$l_q^2(\hbar, E_j^{(l)}, \omega_j) = \frac{2E_j^{(l)}}{\omega_j^2} + \mathrm{O}(\hbar) \tag{5.17}$$

and, therefore, we recover all the results of section 4.4, i.e. the quantity $\sqrt{2E_j^{(l)}}/\omega_j$ (that is assumed to be independent of j) determines the radius of the configuration space in the classical theory. The topology $(\otimes S^1)^m$ of the configuration space remains unchanged in quantum and classical theories.

A quantization of a classical theory possessing a non-trivial and non-degenerate symplectic structure can be performed by two methods. The first is to postulate the commutation relations induced by the Poisson bracket

$$[\hat{X}_j, \hat{P}_k] = i\hbar\{X_j, P_k\}|_{X,P=\hat{X},\hat{P}}. \tag{5.18}$$

The other approach supposes the Darboux variables to be found before quantization and the sequential quantization of the Darboux variables, i.e. $\{X_j, \ P_k\} = \{X_j(x, p), \ P_k(x, p)\}$ if $\{x_j, \ p_k\} = \delta_{jk}$ and †

$$[\hat{X}_j, \hat{P}_k] = [X_j(\hat{x}, \hat{p}), P_k(\hat{x}, \hat{p})] \tag{5.19}$$

if $[\hat{x}_j, \ \hat{p}_k] = i\hbar\delta_{jk}$. In general, both approaches can lead to different quantum theories. Indeed, the use of (5.18) gives the relation (5.12) for creation and destruction operators, while the rule (5.19) produces the commutation relation (4.39) which is different from (5.12). To establish a connection between the two theories (5.18) and (5.19), one should represent the operators (5.18) through $\hat{x}_j$ and $\hat{p}_j$, i.e. introduce the Darboux variables after quantization, which has been done when deriving equation (5.15).

5.3. *The principle of least action*

Equations of motion (5.2) and (5.3) can be obtained from the principle of least action. Let us denote the phase-space variables (X_j, P_j) by θ^A such that $\theta^{2j-1} = X_j$ and $\theta^{2j} = P_j$, $j = 1, 2, \ldots, m$. In this notation, the Poisson bracket (5.10) for two functions $F_{1,2} = F_{1,2}(\theta)$ on the phase space reads

$$\{F_1, F_2\} = \omega^{AB}\partial_A F_1 \partial_B F_2 \tag{5.20}$$

where $\partial_A = \partial/\partial\theta^A$ and non-vanishing contravariant components of the symplectic structure are defined by the equality

$$\omega^{2j-1,2j}(\theta) = -\omega^{2j,2j-1}(\theta) = 1 - \frac{1}{2E_j^{(l)}}\left(P_j^2 + \omega_j^2 X_j^2\right) = 1 - \frac{h_j}{E_j^{(l)}}. \tag{5.21}$$

Solutions to the Hamiltonian equations of motion determine an extremum of the action

$$S_q[\theta] = \int \mathrm{d}t \left(\theta^A \bar{\omega}_{AB}\dot{\theta}^B - H_q(\theta)\right) \tag{5.22}$$

where the functions $\bar{\omega}_{AB}$ are defined in terms of covariant components ω_{AB}, $\omega_{AC}\omega^{CB} = \delta_A^B$, of the symplectic structure [22]

$$\bar{\omega}_{AB}(\theta) = (\theta^C\partial_C + 2)^{-1}\omega_{AB}(\theta) = \int_0^1 \omega_{AB}(\alpha\theta)\alpha\,\mathrm{d}\alpha. \tag{5.23}$$

† See footnote 1 on p 2597.

Substituting the matrix inverse to (5.21),

$$\omega_{2j-1,2j}(\theta) = -\omega_{2j,2j-1}(\theta) = -\left(1 - \frac{h_j}{E_j^{(l)}}\right)^{-1} \tag{5.24}$$

into (5.23) and doing the integral over α we obtain

$$\bar{\omega}_{2j-1,2j}(\theta) = \frac{E_j^{(l)}}{2h_j} \ln\left(1 - \frac{h_j}{E_j^{(l)}}\right). \tag{5.25}$$

To be convinced that the equation $\delta S_q = 0$ is equivalent to Hamiltonian equations of motion (5.2) and (5.3),

$$\dot{\theta}^A = \omega^{AB}(\theta)\partial_B H_q(\theta) \tag{5.26}$$

one should use the standard algebraic property of the symplectic metric [1]

$$\partial_A \omega_{BC} + \partial_B \omega_{CA} + \partial_C \omega_{AB} = 0 \tag{5.27}$$

that results from the Jacobi identity for the Poisson bracket (5.20), for calculating the variation of the action. Notice that the action (5.22) contains the matrix $\bar{\omega}_{AB}$ that satisfies the identity (5.27) too as follows from its definition (5.23).

5.4. *Representation of $U_q(m)$ on a commutative phase space*

A quantum system of m q-deformed oscillators with equal frequencies has the $U_q(m)$ symmetry as is shown in section 3. Therefore the corresponding classical theory must also have this symmetry. It means that the Poisson bracket of a classical Hamiltonian with all the generators of $U_q(m)$ vanishes.

To obtain the generators of $U_q(m)$ as functions on the phase space $(\boldsymbol{X}, \boldsymbol{P})$ (and, hence, on the phase space of the canonical variables $(\boldsymbol{x}, \boldsymbol{p})$), one can simply change the creation and destruction operators in (3.11) and (3.12) written via the operators (3.6) by the corresponding deformed holomorphic variables $\boldsymbol{b}^*$ and $\boldsymbol{b}$ in accordance with the classical limit investigated above. Let us first construct the classical analogue of the operators $\hat{N}_i$. The characteristic properties of the number operator are given by (3.3). So, one should find functions $H_i = H_i(\boldsymbol{X}, \boldsymbol{P})$ such that their Poisson bracket with the holomorphic variables $\boldsymbol{b}_j$ and $\boldsymbol{b}_j^*$ would be proportional to $\boldsymbol{b}_j$ and $\boldsymbol{b}_j^*$, respectively. We define the functions H_i by the following equalities

$$h_i = \omega b_i^* b_i = E^{(l)}(1 - \mathrm{e}^{-H_i/E^{(l)}}) \tag{5.28}$$

where ω is a frequency of all oscillators. The relation (5.28) can be treated as a formal classical limit of the operator equality (2.7). Notice that in the case of usual oscillators, the classical limit means that an eigenvalue of the number operator tends to infinity, $\hat{N}_i \to \infty$, while the operator $\hbar\omega\hat{N}_i$ turns into the classical Hamiltonian of one free oscillator as $\hbar \to 0$ and $\hat{N}_i \to \infty$. Therefore, one can assume H_i to be a classical limit of the operator $\hbar\omega\hat{N}_i$ entering into the equality (2.7) written for each degree of freedom. It is easy to see that the functions H_i have the remarkable properties

$$\{H_i,\ b_j^*\} = -\mathrm{i}\omega\delta_{ij}b_j^* \qquad \{H_i,\ b_j\} = i\omega\delta_{ij}b_j \tag{5.29}$$

which are classical analogies of the relations (3.3) for the operators (3.6).

Now we can determine the formal classical limit of the operators $\hat{a}_j$ and $\hat{a}_j^+$. Changing the operators $\hat{b}_j$ and $\hat{b}_j^+$ by the corresponding holomorphic variables b_j and b_j^* in (3.6) and then taking into account that $\hbar\omega\hat{N}_j \to H_j$ as $\hbar \to 0$, we obtain

$$a_i = b_i \exp(H_i/4E^{(l)}) \qquad a_i^* = b_i^* \exp(H_i/4E^{(l)}) \tag{5.30}$$

as the classical limit of the operators $\hbar^{1/2}\hat{a}_i$ and $\hbar^{1/2}\hat{a}_i^+$, respectively. The Poisson bracket of the new holomorphic variables (5.30) has the form

$$\{a_i,\, a_j^*\} = -\mathrm{i}\delta_{ij} \mathrm{e}^{H_i/(2E^{(l)})}\left(1 - \frac{h_i}{2E^{(l)}}\right) = -\mathrm{i}\delta_{ij}\cosh\frac{H_i}{2E^{(l)}} \tag{5.31}$$

$$\{H_i,\, a_j^*\} = -\mathrm{i}\omega\delta_{ij}a_j^* \qquad \{H_i,\, a_j\} = \mathrm{i}\omega\delta_{ij}a_j \tag{5.32}$$

$$a_i^* a_i = \frac{2E^{(l)}}{\omega}\sinh\frac{H_j}{2E^{(l)}} \tag{5.33}$$

(compare (5.32) with (3.3) and (5.33) with (3.2)). In accordance with the rules of the classical limit established above, we define the generators of $U_q(m)$ as the following functions on the phase space spanned by commutative coordinates $(\boldsymbol{X}, \boldsymbol{P})$

$$\mathrm{e}_{ij} = a_i^* a_j \qquad \mathrm{i} \neq j = 1, 2, \ldots, m \tag{5.34}$$

$$d_j = \frac{1}{\omega}(H_j - H_{j+1}) \qquad j = 1, 2, \ldots, m-1 \tag{5.35}$$

$$d = \frac{1}{\omega}H = \frac{1}{\omega}\sum_{j=1}^{m} H_j. \tag{5.36}$$

The commutation relations for the generators with respect to the Poisson bracket (5.10) (or (5.31)) read

$$\left\{\mathrm{e}_{jk},\, \mathrm{e}_{j'k'}\right\} = -\mathrm{i}\delta_{j'k}\mathrm{e}_{jk'}\cosh\frac{H_k}{2E^{(l)}} + \mathrm{i}\delta_{jk'}\mathrm{e}_{j'k}\,\cosh\frac{H_j}{2E^{(l)}} \tag{5.37}$$

$$\left\{\mathrm{e}_{jk},\, d_n\right\} = -\mathrm{i}(\delta_{kn} - \delta_{jn} - \delta_{kn+1} + \delta_{jn+1})\mathrm{e}_{jk} \tag{5.38}$$

$$\left\{\mathrm{e}_{jk},\, d\right\} = 0 \qquad \{d_j,\, d\} = 0 \tag{5.39}$$

In the case of $j = k'$ or $j' = k$ in the right-hand side of the equality (5.37), the quantity e_{jj} must be treated as $a_j^* a_j$ defined by equation (5.33). The Hamiltonian, being a function of d, obviously commutes with all the generators (5.34)–(5.36). After the canonical quantization of (5.34)–(5.36), we recover the bosonic q-oscillator realization of $U_q(m)$ (3.11)–(3.13).

Thus, the generators (5.34)–(5.36) determine an action of $U_q(m)$ on the phase space spanned by commutative coordinates with respect to the Poisson bracket constructed above and, hence, they realize a representation of $U_q(m)$ in a space of functions on the phase space $(\boldsymbol{X}, \boldsymbol{P})$ (or $(\boldsymbol{x}, \boldsymbol{p})$). This means that we associated a vector field V_σ with each generator $\sigma = (\mathrm{e}_{ij},\, d_i,\, d)$ of $U_q(m)$ such that

$$V_\sigma F = \{\sigma,\, F\} \tag{5.40}$$

for an arbitrary function F on the phase space. The vector fields V_σ realize an action of $U_q(m)$ on the phase space because of the property

$$[V_\sigma, V_{\sigma'}] = V_{\{\sigma,\sigma'\}} \tag{5.41}$$

which is guaranteed by the Jacobi identity for the Poisson bracket.

Concluding this section we would like to emphasize that a system of q-oscillators is not the only q-deformed system which can be described by the 'deformed' symplectic structure. In our recent work [23], we have developed this approach for a q-particle (or a particle on the q-line) [24]. It turns out that the q-particle can be treated as a particle with friction (the friction force acting on the particle is proportional to its velocity). We also construct constrained systems whose dynamics, being reduced on the physical phase space, is equivalent to dynamics of the q-oscillator or the q-particle [23].

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Appendix A. The path integral and boundary conditions

A1. A particle in a box

The transition amplitude $U_t(x,x') = \langle x|\exp(-it\hat{H}/\hbar)|x'\rangle$ for a free particle moving in a one-dimensional box, $\hat{H} = -\hbar^2/2\partial^2/\partial x^2$, satisfies the Schrödinger equation

$$i\hbar\partial_t U_t(x,x') = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2}U_t(x,x') \tag{A1}$$

with the initial condition

$$U_{t=0}(x,x') = \delta(x-x') \qquad x, x' \in [0,l] \tag{A2}$$

and the boundary conditions

$$U_t(0,x') = U_t(l,x') = U_t(x,0) = U_t(x,l) = 0 \tag{A3}$$

where l is a size of the box. The solution to equation (A1) with the initial condition (A2) is well known

$$\tilde{U}_t(x,x') = (2\pi i\hbar t)^{-1/2}\exp[i(x-x')^2/2\hbar t]. \tag{A4}$$

However, it does not satisfy the boundary conditions (A3). To obtain a solution to (A1)–(A3), one should take the following linear combination [14]

$$U_t(x,x') = (2\pi i\hbar t)^{-1/2} \sum_{n=-\infty}^{\infty} \left(\exp \frac{i(x-x'-2ln)^2}{2\hbar t} - \exp \frac{i(x+x'-2ln)^2}{2\hbar t} \right) \tag{A5}$$

$$= \int_{-\infty}^{\infty} dx'' \, \tilde{U}_t(x,x'') Q(x'',x') \tag{A6}$$

$$Q(x'',x') = \sum_{n=-\infty}^{\infty} \left[\delta(x''-x'-2ln) - \delta(x''+x'-2ln) \right]. \tag{A7}$$

The operator $\hat{Q}$ contains all information about the boundary conditions. The kernel (A4) has the standard path integral representation

$$\tilde{U}_t(x,x') = \int_{-\infty}^{\infty} \prod_{\tau=0}^{t} \left(\frac{dp(\tau)dx(\tau)}{2\pi\hbar} \right) \exp \frac{i}{\hbar} \int_0^t d\tau \left(p\dot{x} - \frac{1}{2}p^2 \right) \tag{A8}$$

where the integral is taken over all paths connecting the initial point $x(0) = x'$ with the final one $x(t) = x$. The symmetrization (A6) of the path integral (A8) means that except direct trajectories connecting the points x' and x, one should also include trajectories reflected from the 'walls' into the path sum. The sign of a contribution of a reflected trajectory into the transition amplitude depends on the type of boundary conditions [15].

A2. A particle on a circle

The transition amplitude for a free particle on a circle satisfies the periodic boundary condition

$$U_t(x+l,x') = U_t(x,x'+l) = U_t(x,x') \tag{A9}$$

and the Schrödinger equation (A1) with the initial condition (A2). The path integral representation for the transition amplitude has the form (A6) where $\tilde{U}_t(x,x'')$ is given by (A8), but the operator $\hat{Q}$ has to be changed as follows

$$Q(x,x') = \sum_{n=-\infty}^{\infty} \delta(x''-x'-ln) \tag{A10}$$

because of the other choice of boundary conditions. Equality (A10) shows us that all contributions of the reflected trajectories are taken with the same sign if the configuration space is topologically equivalent to a circle S^1. It is exactly the case of q-deformed systems (see sections 4.2 and (4.7), (4.23) therein). The structure (A6) (and (4.23), as well) of the transition amplitude is preserved in the semiclassical approximation and, therefore, one can state that the configuration space of classical q-deformed systems is $(\otimes S^1)^m$.

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